Goldberg's Ultrapower Axiom

W. Hugh Woodin

Harvard University

ESTS 2024
Münster
September 16, 2024

Beyond the basic axioms: large cardinal axioms

Sharpening the conception of V

- The ZFC axioms are naturally augmented by additional axioms which assert the existence of "very large" infinite sets.
  - Such axioms assert the existence of **large cardinals**.

These large cardinals include:

- Measurable cardinals
- Strong cardinals
- Woodin cardinals
- Superstrong cardinals
- Supercompact cardinals
- Extendible cardinals
- Huge cardinals
- ω-huge cardinals
- Axiom I_0 cardinals

Supercompact cardinals

Suppose $\kappa < \lambda$ are uncountable cardinals.

- $\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \mid |\sigma| < \kappa\}$.
- Suppose U is an ultrafilter on I where $I = \mathcal{P}_\kappa(\lambda)$.
  - U is **κ-complete** if U is closed under intersections of cardinality less that κ.
  - U is **fine** if for all $\alpha < \lambda$, $\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in \sigma\} \in U$.
  - U is **normal** if for all $F : \mathcal{P}_\kappa(\lambda) \to \lambda$, if
  $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid F(\sigma) \in \sigma\} \in U$$
  then there exists $X \in U$ such that $F|X$ is constant.

Definition (Reinhardt,Solovay:1967)

Suppose κ is an uncountable cardinal.

- Then κ is **supercompact** if for all $\lambda > \kappa$ there is a κ-complete normal fine ultrafilter U on $\mathcal{P}_\kappa(\lambda)$.

Strongly compact cardinals

Definition (Keisler-Tarski:1963)

Suppose that κ is an uncountable regular cardinal. Then κ is a **strongly compact cardinal** if for each $\lambda > \kappa$ there exists an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that:

1. U is a κ-complete ultrafilter,
2. U is a fine ultrafilter.

- One is just dropping the normality requirement.

Theorem (Menas:1976)

Suppose κ is a measurable cardinal and that κ is a limit of strongly compact cardinals.

- *Then κ is a strongly compact cardinal.*

- Every supercompact cardinal is a strongly compact cardinal.
- The Menas Theorem shows the converse can naturally fail:
  - The least measurable cardinal which is a limit of supercompact cardinals is **not** a supercompact cardinal.

Solovay's conjecture

Conjecture (Solovay)

The following are equiconsistent.

1. ZFC + "There is a supercompact cardinal".
2. ZFC + "There is a strongly compact cardinal".

- This is one of the central problems of the Inner Model Program.

The Menas Theorem leaves open the possibility that the following might be equivalent.

1. *There is a supercompact cardinal.*
2. *There is a strongly compact cardinal.*

The Identity Crisis Theorem of Magidor

Lemma

Suppose that κ is a supercompact cardinal. Then

- *κ is a limit of measurable cardinals.*

Theorem (Magidor:1976)

Suppose κ is a strongly compact cardinal. Then there is a (class) generic extension of V in which:

- *κ is a strongly compact cardinal.*
- *κ is the **only** measurable cardinal.*

As a consequence:

- Solovay's Conjecture looks extremely difficult to solve.

Conjecture (Magidor)

*The following are **not** equiconsistent.*

1. ZFC + "There is a supercompact cardinal".
2. ZFC + "There is a strongly compact cardinal".

Close embeddings and finitely generated models

Definition

Suppose that M, N are transitive sets, $M \models \mathrm{ZFC}$, and that

$$\pi : M \to N$$

is an elementary embedding. Then π is **close** to M if for each $X \in M$ and each $a \in \pi(X)$,

$$\{Z \in \mathcal{P}(X) \cap M \mid a \in \pi(Z)\} \in M.$$

Definition

Suppose that N is a transitive set such that

$$N \models \mathrm{ZFC} + \text{“}V = \mathrm{HOD}\text{”}.$$

Then N is **finitely generated** if there exists $a \in N$ such that every element of N is definable in N from a.

Why close embeddings?

Lemma

Suppose that M is a transitive set,

$$M \models \mathrm{ZFC} + \text{“}V = \mathrm{HOD}\text{”},$$

and that M is finitely generated.

- *Suppose that N is a transitive set and*
  - $\pi_0 : M \to N$
  - $\pi_1 : M \to N$

  are elementary embeddings each of which is close to M.
- *Then $\pi_0 = \pi_1$.*

- Without the requirement of closeness, the conclusion that $\pi_0 = \pi_1$ can fail.

Weak Comparison

Definition

Suppose that $V = \mathrm{HOD}$. Then **Weak Comparison** holds if for all $X, Y \prec_{\Sigma_2} V$ the following hold where M_X is the transitive collapse of X and M_Y is the transitive collapse of Y.

- Suppose that M_X and M_Y are finitely generated models of ZFC, $M_X \neq M_Y$, and
  - $M_X \cap \mathbb{R} = M_Y \cap \mathbb{R}$.
- Then there exists a transitive set M^* and elementary embeddings
  - $\pi_X : M_X \to M^*$
  - $\pi_Y : M_Y \to M^*$

  such that π_X is close to M_X and π_Y is close to M_Y.

- Weak Comparison holds in all the inner models which have been constructed in the Inner Model Program.
  - It is a simple consequence of the incredible structure these models have.

Goldberg's Ultrapower Axiom

Notation

Suppose that $N \models \mathrm{ZFC}$ is an inner model of ZFC, $U \in N$ and

$$N \models \text{"}U \text{ is a countable complete ultrafilter"}$$

- N_U denotes the transitive collapse of $\mathrm{Ult}_0(N, U)$
- $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Definition (**The Ultrapower Axiom**)

Suppose that U and W are countably complete ultrafilters. Then there exist $W^* \in V_U$ and $U^* \in V_W$ such that the following hold.

(1) $V_U \models$ "W^* is a countable complete ultrafilter".

(2) $V_W \models$ "U^* is a countable complete ultrafilter".

(3) $(V_U)_{W^*} = (V_W)_{U^*}$.

(4) $j_{W^*}^{V_U} \circ j_U^V = j_{U^*}^{V_W} \circ j_W^V$.

- If $V = \mathrm{HOD}$ then (3) implies (4).

Weak Comparison and the Ultrapower Axiom

- The Ultrapower Axiom simply asserts that amalgamation holds for the ultrapowers of V by countably complete ultrafilters.
- If there are no measurable cardinals then the Ultrapower Axiom holds trivially
  - since every countably complete ultrafilter is principal.

Theorem (Goldberg)

Suppose that $V = \mathrm{HOD}$ and that there exists

$$X \prec_{\Sigma_2} V$$

such that $M_X \models \mathrm{ZFC}$ where M_X is the transitive collapse of X. Suppose that Weak Comparison holds.

- *Then the Ultrapower Axiom holds.*

- If X does not exist then Weak Comparison holds vacuously.
- Assuming large cardinals exist then X must exist.

The Ultrapower Axiom and strongly compact cardinals

Theorem (Goldberg)

Assume the Ultrapower Axiom and that for some κ:

- *κ is a strongly compact cardinal.*
- *κ is not a supercompact cardinal.*

Then κ is a measurable limit of supercompact cardinals.

- The Ultrapower Axiom resolves the "identity crisis".
  - By the Menas Theorem, this resolution is best possible.

Corollary (Goldberg)

The following are equiconsistent, and in fact equivalent.

1. $\mathrm{ZFC} + \mathrm{UA} +$ "There is a supercompact cardinal".
2. $\mathrm{ZFC} + \mathrm{UA} +$ "There is a strongly compact cardinal".

The power of the Ultrapower Axiom

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is supercompact. Then

- *Suppose $A \subset \kappa$ codes V_κ. Then $V = \mathrm{HOD}_A$.*
  - *V is a generic extension of HOD.*
- *GCH holds at all cardinals $\gamma \geq \kappa$.*

Theorem (Goldberg)

Assume the Ultrapower Axiom. Then the following are equivalent.

1. *There is a supercompact cardinal.*
2. *There is a cardinal κ such that for all λ, there is a countably complete ultrafilter U such that $j_U(\kappa) > \lambda$ where*
$$j_U : V \to M_U$$
is the ultrapower embedding.

Descriptive Set Theory: Prewellorderings and scales

Definition (ZF)

A preorder $\leq$ on $A \subseteq \mathbb{R}$ is a **prewellordering** if every nonempty subset of A has a $\leq$-least element.

- A prewellorder on A is simply an equivalence relation on A together with a wellordering of the equivalence classes.

Definition (ZF)

(**Moschovakis:1971**) Suppose $A \subseteq \mathbb{R}$. A **scale** on A is a sequence

$$\langle \leq_i : i < \omega \rangle$$

of prewellorderings on A such that the following hold.

1. For all $x, y \in A$, for all $i < \omega$, if $x \leq_{i+1} y$ then $x \leq_i y$.
2. Suppose $\langle \sigma_k : k < \omega \rangle$ is an infinite sequence of nonempty subsets of A, with limit $x \in \mathbb{R}$, such that
   - For all $i < \omega$, $y \sim_i z$ for all $y, z \in \cup_{k \geq i} \sigma_k$.

   Then $x \in A$ and for all $i < \omega$, $x \leq_i y$ for all $y \in \cup_{k \geq i} \sigma_k$.

Beyond the Borel sets: The Universally Baire sets

Definition (Feng-Magidor-Woodin:1991)

A set $A \subseteq \mathbb{R}$ is **universally Baire** if:

- For all topological spaces Ω
- For all continuous functions $\pi : \Omega \to \mathbb{R}$;

the preimage of A by π has the property of Baire in the space Ω.

- Every Borel set is universally Baire.

Lemma

Suppose $A \subseteq \mathbb{R}$ is universally Baire. Then A is Lebesgue measurable and has the property of Baire.

- It is consistent with ZFC that every set $A \subseteq \mathbb{R}$ is the image of a universally Baire set by a continuous function $f : \mathbb{R} \to \mathbb{R}$.
  - For example, this holds if $V = L$.

The influence of large cardinals

- Universally Baire subsets of $\mathbb{R} \times \mathbb{R}$ are defined in exactly the same way as the universally Baire subsets of $\mathbb{R}$.

Theorem

Assume there is a proper class of Woodin cardinals and that A is universally Baire. Then the following hold.

1. (Woodin) *Every set $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire.*
2. (Steel) *A has a universally Baire scale.*
3. (Martin, Steel) *A is determined.*

Transfinite Borel sets

${}^{\infty}$Borel Codes

- All increasing pairs of rational numbers, are ${}^{\infty}$-Borel codes.
- If S is an ${}^{\infty}$-Borel code then $(0, S)$ is an ${}^{\infty}$-Borel code.
- A transfinite sequence, $\langle S_\alpha : \alpha < \eta \rangle$, is an ${}^{\infty}$-Borel code if S_α is an ${}^{\infty}$-Borel code for all $\alpha < \eta$.

The interpretation of an ${}^{\infty}$Borel Code S as a set $A_S \subseteq \mathbb{R}$

- If $S \in \mathbb{Q} \times \mathbb{Q}$ then A_S is the interval $[r, s]$
- If $S = (0, T)$ then $A_S = \mathbb{R} \backslash A_T$.
- If $S = \langle S_\alpha : \alpha < \eta \rangle$ then $A_S = \bigcup_{\alpha < \eta} A_{S_\alpha}$.

- A set $X \subseteq \mathbb{R}$ is ${}^{\infty}$-Borel if $X = A_S$ for some ${}^{\infty}$-Borel code, S.

$^\infty$Borel sets without the Axiom of Choice

- Assuming the Axiom of Choice, every set $X \subseteq \mathbb{R}$ is $^\infty$Borel.
  - One cannot prove in ZF that even all the Σ^1_3-sets are $^\infty\mathrm{Borel}$.

Lemma (ZF)

Suppose $A \subseteq \mathbb{R}$ and there is a scale on A.

- *Then A is $^\infty$Borel.*

Lemma (ZF)

Assume $A \subseteq \mathbb{R}$ is $^\infty$Borel and that there is no uncountable set $X \subseteq \mathbb{R}$ such that X can be wellordered.

- *Then A is Lebesgue measurable and has the property of Baire.*

A technical refinement of AD

Definition

AD^+ **holds** if:

- Suppose there is a prewellordering of $\mathbb{R}$ of length θ and
$$\pi : \theta^\omega \to \omega^\omega$$
is continuous. Then
  - For each set $A \subseteq \omega^\omega$, the set $\pi^{-1}[A] \subseteq \theta^\omega$ is determined:
    - where in this game, the players alternate choosing ordinals α such that $\alpha < \theta$.
- Every set $A \subseteq \mathbb{R}$ is $^\infty$Borel.

- AD^+ implies AD
  - Just use the identity function $\pi : \omega^\omega \to \omega^\omega$.
- (Conjecture) AD implies AD^+.

The universally Baire sets and AD^+

Lemma (Solovay)

Suppose $A \subseteq \mathbb{R}$. Then the following are equivalent.

1. *There is a wellordering of $\mathbb{R}$ in $L(A, \mathbb{R})$.*
2. *For every set $B \subseteq \mathbb{R}$, if $B \in L(A, \mathbb{R})$ then B has a scale in $L(A, \mathbb{R})$.*

- The equivalence fails if one just requires that B is ${}^{\infty}$Borel in $L(A, \mathbb{R})$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire. Then

$$L(A, \mathbb{R}) \models \mathrm{AD}^+.$$

- $L(\mathbb{R}) \models \mathrm{AD}$ if and only if $L(\mathbb{R}) \models \mathrm{AD}^+$.

HOD in AD^+ models

The first connection of AD with large cardinals:

Theorem (Solovay)

Suppose $A \subseteq \mathbb{R}$ and that $L(A, \mathbb{R}) \models \mathrm{AD}$. Then ω_1 is a measurable cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

Theorem

Suppose $A \subseteq \mathbb{R}$ and that $L(A, \mathbb{R}) \models \mathrm{AD}$. Let

- *$\Theta^{L(A,\mathbb{R})}$ be the supremum of the lengths of all prewellorderings of $\mathbb{R}$ which belong to $L(A, \mathbb{R})$.*

Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

Theorem

Suppose $A \subseteq \mathbb{R}$ and that $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then ω_1 is the least measurable cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

This motivates the natural conjecture that if $L(A, \mathbb{R}) \models \mathrm{AD}^+$ then

- $\mathrm{HOD}^{L(A,\mathbb{R})}$ is a "canonical model".

The Inner Model Program

Theorem (Scott:1961)

Assume $V = L$. Then there are no measurable cardinals.

- The **Inner Model Program** seeks to construct enlargements of L in which large cardinals can exist.
  - These enlargements are **core models**.
  - The stronger the large cardinal notion the harder the problem.

A remarkable convergence and a surprise (1988-96)

Assume $\mathrm{AD}^{L(\mathbb{R})}$ and let Θ be the supremum of the lengths of the prewellorderings in $L(\mathbb{R})$.

- (Steel) $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$ is a core model.
- (Woodin) $\mathrm{HOD}^{L(\mathbb{R})}$ is **not** a core model,
  - it is a **strategic-core model**.

- A new class of enlargements of L is naturally revealed by AD^+
  - strategic-core models.

The axiom $V = \mathrm{Ultimate}\text{-}L$

The axiom for $V = \mathrm{Ultimate}\text{-}L$

- There is a proper class of Woodin cardinals.
- For each Σ_2-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that
$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi.$$

Theorem

Assume $V = \mathrm{Ultimate}\text{-}L$. Then the following hold.

1. CH.
2. $V = \mathrm{HOD}$.
3. *V is not a generic extension of any inner model.*

Scales and Suslin cardinals

Definition

Suppose $A \subseteq \mathbb{R}$ and λ is an infinite cardinal. Then A is λ-**Suslin** if there is a scale on A with associated prewellorderings of length at most λ.

Definition

Suppose λ is an infinite cardinal. Then λ is a **Suslin cardinal** if there exists a set $A \subseteq \mathbb{R}$ such that

- A is λ-Suslin.
- A is not γ-Suslin for any $\gamma < \lambda$.

- (ZF) ω and ω_1 are Suslin cardinals.

AD^+ and Suslin cardinals

Theorem

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}$. Then the following are equivalent.

1. $L(A, \mathbb{R}) \models \mathrm{AD}^+$.
2. $L(A, \mathbb{R}) \models$ "There is a largest Suslin cardinal".

- This theorem is one of the many equivalences of AD^+ in the context of AD, which have emerged over that last 30 years.

The largest Suslin cardinal in $L(A, \mathbb{R})$

Notation

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then

- δ_A is the largest Suslin cardinal of $L(A, \mathbb{R})$.
- $\Theta_A = \Theta^{L(A,\mathbb{R})}$.

Theorem

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then

- *δ_A is strongly inaccessible in $\mathrm{HOD}^{L(A,\mathbb{R})}$.*
- $\mathrm{HOD}^{L(A,\mathbb{R})} \restriction \delta_A \prec_{\Sigma_2} \mathrm{HOD}^{L(A,\mathbb{R})} \restriction \Theta_A$.

More notation

- $H_A = \mathrm{HOD}^{L(A,\mathbb{R})} \restriction \delta_A$.

- $H_A \models \mathrm{ZFC}$.

LSA models

Definition

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then $L(A, \mathbb{R})$ is an **LSA model** if for all $\gamma < \delta_A$, if

$$\pi : \mathcal{P}(\gamma) \cap L(A, \mathbb{R}) \to \delta_A$$

is a function such that $\pi \in L(A, \mathbb{R})$ and such that π is OD in $L(A, \mathbb{R})$, then the range of π is bounded.

Theorem

Suppose that $A \subseteq \mathbb{R}$, $L(A, \mathbb{R}) \models \mathrm{AD}^+$, and that $L(A, \mathbb{R})$ is an LSA *model. Then*

$$H_A \models \mathrm{ZFC} + \text{“}V = \mathrm{HOD}\text{”}$$

- It is conjectured that one can drop the requirement that $L(A, \mathbb{R})$ be an LSA model.

LSA models and the Ultrapower Axiom

Theorem

Suppose that $A \subseteq \mathbb{R}$, $L(A, \mathbb{R}) \models \mathrm{AD}^+$, *and that* $L(A, \mathbb{R})$ *is an* LSA *model. Then*

$$H_A \models \mathrm{ZFC} + \text{Weak Comparison}.$$

- Thus by Goldberg's Theorem:

Theorem

Suppose that $A \subseteq \mathbb{R}$, $L(A, \mathbb{R}) \models \mathrm{AD}^+$, *and that* $L(A, \mathbb{R})$ *is an* LSA *model. Then*

$$H_A \models \mathrm{ZFC} + \text{Ultrapower Axiom}.$$

- But what about $\mathrm{HOD}^{L(A,\mathbb{R})}$?

H_A versus $\mathrm{HOD}^{L(A,\mathbb{R})}|\Theta_A$

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then

- $(H_A =)\,\mathrm{HOD}^{L(A,\mathbb{R})}|\delta_A \prec_{\Sigma_2} \mathrm{HOD}^{L(A,\mathbb{R})}|\Theta_A$.

As a corollary, using Goldberg's analysis of the Ultrapower Axiom:

Theorem

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then the following are equivalent.

1. $H_A \models \text{Ultrapower Axiom}$.
2. $\mathrm{HOD}^{L(A,\mathbb{R})} \models \text{Ultrapower Axiom}$.

Theorem

Suppose that $A \subseteq \mathbb{R}$, $L(A, \mathbb{R}) \models \mathrm{AD}^+$, and that $L(A, \mathbb{R})$ is an LSA *model. Then*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \text{Ultrapower Axiom}.$$

The general case

Notation

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$.

- T_A denotes the Σ_1-theory of $L(A, \mathbb{R})$ with parameters from $\delta_A \cup \{\mathbb{R}\}$.

Theorem

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then (in the language of Set Theory with an additional predicate)

- $(H_A, T_A) \models \mathrm{ZFC} + \text{“}V = \mathrm{HOD}\text{”}$.
- $(H_A, T_A) \models \mathrm{ZFC} + \mathrm{Weak\ Comparison}$.

Theorem

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \mathrm{ZFC} + \mathrm{Ultrapower\ Axiom}.$$

$V = \mathrm{Ultimate}\text{-}L$ and the Ultrapower Axiom

Theorem (Goldberg)

The following are equivalent.

1. Ultrapower Axiom.
2. *For all $\gamma > \omega$, if $\gamma = |V_\gamma|$ then*

$$V_\gamma \models \text{Ultrapower Axiom}.$$

- Thus the negation of Ultrapower Axiom is expressible by a Σ_2-sentence
  - which cannot reflect into $\mathrm{HOD}^{L(A,\mathbb{R})}$.

Theorem

Assume $V = \mathrm{Ultimate}\text{-}L$. Then the Ultrapower Axiom *holds.*

A deeper connection?

Definition (Hamkins)

1. An inner model N is a **ground** if $V = N[G]$.
2. The **mantle** of V is the intersection of all the grounds of V.
3. **Ground Axiom**: The only ground of V is V.

Theorem (Usuba)

Suppose there is an extendible cardinal and that $\mathbb{M}$ is the mantle of V. Then $\mathbb{M}$ is a ground of V.

Mantle Conjecture

Assume there is an extendible cardinal and that

$$V \models \text{Ultrapower Axiom}.$$

Then $\mathbb{M} \models$ "$V = \text{Ultimate-}L$".

- The Mantle Conjecture implies (assuming there is an extendible cardinal) that the axiom $V = \text{Ultimate-}L$ is **equivalent** to:
  - Ultrapower Axiom + Ground Axiom.

The Ultimate-L Program

One central goal of the Ultimate-L Program is to prove the following conjecture.

- This would also likely achieve many of the current goals of the Inner Model Program.

Conjecture

Suppose that $A \subseteq \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then

- $\mathrm{HOD}^{L(A,\mathbb{R})}$ *is a strategic-core model.*

The theorem that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \text{Ultrapower Axiom}$$

confirms that Goldberg's Ultrapower Axiom will play a key role in the Ultimate-L Program.