A lightning introduction to homological algebra and condensed mathematics

Seventh European Set Theory Colloquium

12 February 2026

EUROPEAN
SET THEORY
SOCIETY

Homological algebra

Homological algebra is the general algebraic study of *homology*, which in turn is (among other things) a method for encoding information about mathematical structures (especially topological spaces) in algebraic objects.

Abelian categories

Abelian categories provide ideal settings in which to do homological algebra. Roughly speaking, an abelian category is an additive category in which kernels and cokernels of maps exist and behave as desired. The prototypical example of an abelian category is Ab, the category of abelian groups.

Exact sequences

Exact sequences form an important tool in homological algebra. A pair of morphisms

$$A \xrightarrow{\pi} B \xrightarrow{\sigma} C$$

in an abelian category is *exact* at B if $\ker(\sigma) = \mathrm{im}(\pi)$. A *short exact sequence* is a sequence

$$0 \to A \xrightarrow{\pi} B \xrightarrow{\sigma} C \to 0$$

that is exact at A, B, and C.

Exact functors

A functor between abelian categories is said to be *exact* if it preserves short exact sequences. Many standard functors fail to be exact, e.g., the contravariant functor $\mathrm{Hom}(\cdot, \mathbb{Z}) : \mathrm{Ab} \to \mathrm{Ab}$:

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

is exact in Ab, but the induced sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0$$

fails to be exact at the rightmost $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$.

Derived functors

The failure of functors to be exact can often be measured by *derived functors*. For example, the derived functors of Hom are denoted by Ext^i for $i > 0$. Given a short exact sequence $0 \to A \to B \to C \to 0$ in an abelian category $\mathcal{A}$ and an arbitrary $H \in \mathcal{A}$, we obtain a *long* exact sequence

$$0 \longrightarrow \operatorname{Hom}(C,H) \longrightarrow \operatorname{Hom}(B,H) \longrightarrow \operatorname{Hom}(A,H) \longrightarrow$$

$$\hookrightarrow \operatorname{Ext}^1(C,H) \longrightarrow \operatorname{Ext}^1(B,H) \longrightarrow \operatorname{Ext}^1(A,H) \longrightarrow$$

$$\hookrightarrow \operatorname{Ext}^2(C,H) \longrightarrow \operatorname{Ext}^2(B,H) \longrightarrow \operatorname{Ext}^2(C,H) \longrightarrow \dots$$

Whitehead's problem

Nonzero derived functors are manifestly instances of *incompactness*, and especially when the objects involved become uncountable, set theory often has a lot to say about such matters. A prominent early application of set theory to homological algebra is Shelah's solution to Whitehead's problem.

Question (Whitehead)

Suppose that A is an abelian group and $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$. Must A be free?

Theorem (Shelah)

- *If $V = L$, then "yes".*
- *If $\mathrm{MA}_{\aleph_1}$ holds, then "no".*

Condensed mathematics

When one attempts to apply the tools of homological algebra to the study of algebraic objects carrying topologies, one runs into the obstacle that classical categories of such objects are not well-behaved algebraically (e.g., they are typically not abelian). For example, in the category TopAb of topological abelian groups, the identity map

$$(\mathbb{R}, \text{discrete topology}) \to (\mathbb{R}, \text{Euclidean topology})$$

is not an isomorphism, but this is not witnessed by a nontrivial kernel or cokernel. Condensed mathematics seeks a uniform solution to this problem by embedding these classical categories in richer, more well-behaved "condensed" categories.

Condensed categories

A *condensed set/abelian group/ring/...* is (roughly) a contravariant functor $T\colon \text{CHaus} \to \text{Set/Ab/Ring/}\dots$ such that

- $T(\emptyset) = *$;
- for all $S_0, S_1 \in \text{CHaus}$, $T(S_0 \sqcup S_1) = T(S_0) \times T(S_1)$;
- for all surjections $S_1 \twoheadrightarrow S_0$ in CHaus with fiber product $S_1 \times_{S_0} S_1$ and projection maps $\pi_0, \pi_1 : S_1 \times_{S_0} S_1 \to S_1$, the natural map

  $$T(S_0) \to \{x \in T(S_1) \mid T(\pi_0)(x) = T(\pi_1)(x)\}$$

  is a bijection.

Condensed abelian groups

The category Cond(Ab) of condensed abelian groups is a (very nice) abelian category. Moreover, given a (T_1) topological abelian group X, we can form its associated condensed abelian group $\underline{X} \in$ Cond(Ab) by setting $\underline{X}(S) = \mathrm{Cont}(S, X)$ for all $S \in$ CHaus. This defines a functor from TopAb to Cond(Ab). When restricted to, e.g., the category of locally compact topological abelian groups, this functor is fully faithful.