DEFINABLE CARDINALITY

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In this talk I will discuss the concept of definable cardinality for subsets and quotient spaces of the reals (or any Polish space) and compare it with the classical notion of cardinality.

To keep things simple, I will assume here that <u>definable</u> means <u>Borel (definable)</u>.

Let's consider first Borel sets X, Y in Polish spaces. Then X, Y have the same cardinality

$$|X| = |Y|$$

if there is a bijection $f: X \rightarrowtail\!\!\!\twoheadrightarrow Y$.

Since we are in the category of definable sets, it is natural to also consider definable bijections: We say that X, Y have the same <u>Borel cardinality</u>

$$|X|_B = |Y|_B,$$

if there is a Borel bijection $f: X \rightarrowtail\!\!\!\twoheadrightarrow Y$. By classical facts these notions coincide:

$$|X| = |Y| \text{ iff } |X|_B = |Y|_B.$$

The situation though is dramatically different when we study definable quotient spaces

X/E, where X is a Polish space and E a Borel equivalence relation. Since the case of countable X is trivial, I will assume from now on that X is uncountable (thus $|X| = 2^{\aleph_0}$) and, to keep things focused, I will consider the case where E is countable, i.e., every E-class is countable. We say in this case that E is a <u>CBER</u>.

There is a great number of important CBERs appearing in many areas of mathematics, for example:

- Turing or arithmetical equivalence on $2^{\mathbb{N}}$,

- Equality mod finite in $\mathcal{P}(\mathbb{N})$,
- isomorphism of tfa groups of finite rank (i.e., subgroups of $(\mathbb{Q}^n, +)$, for some n),
- orbit equivalence relations induced by Borel actions of a countable group on a Polish space

(In fact by a theorem of Feldman–Moore all CBERs can be generated in this way.)

Given two CBER quotients X/E, Y/F, we say as usual that they have the <u>same cardinality</u>

$$|X/E| = |Y/F|,$$

if there is a bijection $f: X/E \rightarrowtail\!\!\twoheadrightarrow Y/F$.
We say that X/E, Y/F have the <u>same Borel cardinality</u>

$$|X/E|_B = |Y/F|_B,$$

if there is a bijection $f: X/E \rightarrowtail\!\!\twoheadrightarrow Y/F$, which has a Borel lifting $f^*: X \to Y$ (i.e., $xEy \Leftrightarrow f^*(x) F f^*(y)$ and $f([x]_E) = [f^*(x)]_F$. Similarly we define the order

$$|X/E|_B \le |Y/F|_B,$$

if there is such an injection $f: X/E \rightarrowtail Y/F$.

Now it is clear that all X/E, E a CBER, have the same cardinality $|X/E| = 2^{\aleph_0}$, so the picture of cardinality of such quotient spaces is an unispiring dot:

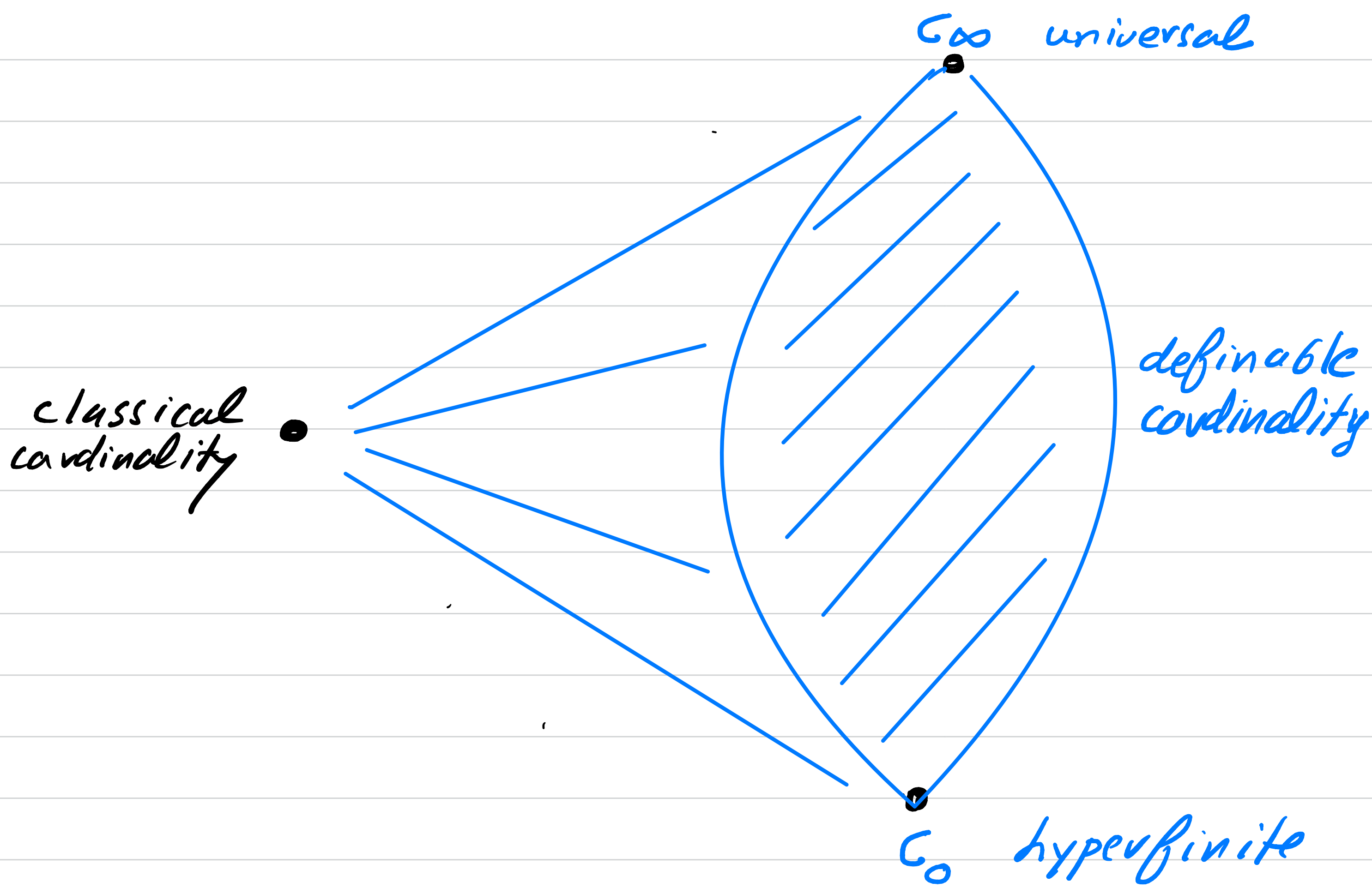
- $|X/E| = 2^{\aleph_0}$

This uses crucially AC and it is a very crude way to measure the size of such quotient spaces, as it does not take into account the structure of E itself. For example, do we really think that there are as many Turing degrees as sets of integers mod finite or that all the orbit spaces of actions of $\mathbb{F}_2$ have the same size as the orbit spaces of actions of $\mathbb{Z}$?

It turns out that studying the <u>Borel cardinality</u> of quotients by CBERs reveals a deep and complex structure.

To discuss this, let me first discard a trivial case: the CBERs E that admit a Borel selector—also called <u>smooth</u>. For those trivially $|X/E|_B = |X/E| = 2^{\aleph_0}$. So from now on all CBERs will be <u>non-smooth</u>.

Here then is a rough picture of definable cardinalities $|X/E|_B$ for CBER E:



• There is a smallest cardinality c_0, which is that of $|X/E|_B$ with E hyperfinite – so all hyperfinite quotient spaces have the same definable cardinality (Dougherty–Jackson–K). Typical examples are

$\mathcal{P}(\mathbb{N})/\text{fin}$, $2^{\mathbb{Z}}/\text{shift}$, $\cong$ of tfa of rank 1.

- There is a largest cardinality c_∞ (DJK), typical examples of which are $2^{\mathbb{F}_2}/\text{shift}$ (DJK), arithmetical equivalence (Slaman–Steel).

- $c_0 < c_\infty$ (follows from classical ergodic theory)

- There is a vast number ($2^{\aleph_0}$ many) of intermediate cardinalities

$$c_0 < c < c_\infty$$

including many incomparable ones (Adams–K). Typical examples are many orbit equivalence relations of <u>free</u> Borel actions of countable groups, including the free actions of non-amenable groups that admit an invariant Borel probability measure, for example the free part of the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$, isomorphism of tfa groups of finite rank > 1 (Hjorth, S. Thomas).

I would like to discuss next some important methodological points:

(A) These results about the definable cardinality structure of X/E, E a CBER, have been proved using methods of ergodic theory.

By contrast generically all CBER are hyperfinite, so on comeager sets their quotient spaces have the same definable cardinality c_0 (Hjorth-K).

Therefore:

(B) It has been a major open problem to find purely set theoretic methods to prove such results.

(C) There are important rigidity phenomena that underlie many of these results. For example, under certain circumstances, if E is the orbit equivalence relation induced by a Borel action of a countable group, then $|X/E|_B$ "remembers" a lot about the group.

For instance, if for a set of odd primes S we let

$$H_S = \ast_{p\in S} (\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \times \mathbb{Z}$$

and E_S is the equivalence relation induced by the shift action of H_S on 2^{H_S}, restricted to its free part, then

$$|X/E_S|_B \le |X/E_T|_B \iff S \subseteq T$$

(Hjorth-K).

For another such situation, if E_n is the isomorphism relation of tfa groups of rank n, then $|X/E_n|_B$ remembers n:

$$|X/E_n|_B \leq |X/E_m|_B \iff n \leq m$$

(S. Thomas). In particular, for $n \geq 2$

$$c_0 < |X/E_n|_B < c_\infty .$$

(D) As seen above:
it should be emphasized here that the existence of intermediate or incomparable cardinalities is not due to the construction of artificial or pathological counterexamples (compare this with incomparable r.e. degrees) but reflects structural differences of natural and important examples.

And let me finish by mentioning a few major open problems that have been open for decades:

(1) What is the definable cardinality c_{TD} of the set of Turing degrees. It is known that

$$c_0 < c_{TD} \leq c_\infty$$

(Slaman-Steel) but whether $c_{TD} = c_\infty$ is unknown (this would contradict Martin's Conjecture on functions on Turing degrees).

(2) Let E be the orbit equivalence relation given by a Borel action

of an <u>amenable</u> group. Is

$|X/E|_B = c_0$, i.e., is E hyperfinite?

(B. Weiss).

(3) Let $E_1 \subseteq E_2 \subseteq \dots$ be

hyperfinite CBERs. Thus $|X/E_n|_B = c_0$.

Let $E = \bigcup_n E_n$. Is $|X/E|_B = c_0$?

(4) If $F \subseteq E$ and $|X/F|_B = c_\infty$

is it true that $|X/E| = c_\infty$? (Hjorth)

THANK YOU!