Exotic Models

Problem

Suppose $\operatorname{ZFC} + \varphi$ has a unique transitive model *M*. Must

$$\blacktriangleright M \models "V = L"?$$

Problem

Suppose $\mathrm{ZFC} + \varphi$ has a unique transitive model *M*. Must

$$\blacktriangleright M \models "V = L"?$$

Theorem (joint with Koellner)

Assume there is a Woodin cardinal with an inaccessible above. There is a formula φ such that the following holds.

For all $x \in \mathbb{R}$, if $M \models \text{ZFC} + \varphi[x]$ and M is transitive, then M is unique and

 $\blacktriangleright M \models "V \neq L[x]".$

Problem

Suppose $ZFC + \varphi$ has a unique transitive model *M*. Must $M \models "V = L"?$

Theorem (joint with Koellner)

Assume there is a Woodin cardinal with an inaccessible above. There is a formula φ such that the following holds.

For all $x \in \mathbb{R}$, if $M \models \text{ZFC} + \varphi[x]$ and M is transitive, then M is unique and

 $\blacktriangleright M \models "V \neq L[x]".$

For a Turing cone of x, there is transitive set M such that

 $\blacktriangleright M \models \text{ZFC} + \varphi[x].$

Problem

Suppose $ZFC + \varphi$ has a unique transitive model *M*. Must $M \models "V = L"?$

Theorem (joint with Koellner)

Assume there is a Woodin cardinal with an inaccessible above. There is a formula φ such that the following holds.

For all $x \in \mathbb{R}$, if $M \models \text{ZFC} + \varphi[x]$ and M is transitive, then M is unique and

 $\blacktriangleright M \models "V \neq L[x]".$

For a Turing cone of x, there is transitive set M such that

 $\blacktriangleright M \models \text{ZFC} + \varphi[x].$

• The proof is nowhere close to solving the problem.

Exotic MM^{++} models

Problem

Are any of the following consistent with $\rm MM^{++}$ and the existence of an extendible cardinal?

- 1. Axiom $(*)^{++}$.
- 2. HOD \models "V = Ultimate-L".
- 3. There is no inner model N of ZFC with the ω_2 -approximation property and the ω_2 -cover property such that ω_2 is strongly inaccessible in N.

Exotic MM^{++} models

Problem

Are any of the following consistent with $\rm MM^{++}$ and the existence of an extendible cardinal?

- 1. Axiom $(*)^{++}$.
- 2. HOD \models "V = Ultimate-L".
- 3. There is no inner model N of ZFC with the ω_2 -approximation property and the ω_2 -cover property such that ω_2 is strongly inaccessible in N.

Theorem

Suppose that δ is a super-Reinhardt cardinal. Then there is a generic extension of V[G] of V such that the following that

• $V[G]_{\delta} \models \text{ZFC} + \text{MM}^{++} + \text{"There is an extendible cardinal"}$ and such that (3) holds in $V[G]_{\delta}$.

But maybe the theorem is vacuously true.

Revised HOD Conjecture

If super-Reinhardt cardinals are consistent then:

► The HOD Conjecture is false.

Problem (Revised HOD Conjecture)

Suppose that δ is an extendible cardinal. Show that one of the following ${\rm must}$ hold.

- 1. HOD is a weak extender model of δ is supercompact.
 - Equivalently, HOD is close to V as in
 - ► The HOD Dichotomy Theorem.
- 2. HOD has no supercompact cardinals.

The HOD-analysis from AD^+ -theory

Problem

Assume
$$A \subseteq \mathbb{R}$$
 and that $L(A, \mathbb{R}) \models AD^+$. Show that
 $\blacktriangleright HOD^{L(A,\mathbb{R})} \models GCH.$

- It seems quite plausible that this is the test question for solving the general HOD-analysis problem of AD⁺.
 - This is the problem of showing that HOD is a "fine-structure model".
 - This problem in turn is deeply connected to the inner model problem for supercompact cardinals.

The HOD-analysis from $\mathrm{AD}^+\text{-}\mathsf{theory}$

Problem

Assume
$$A \subseteq \mathbb{R}$$
 and that $L(A, \mathbb{R}) \models AD^+$. Show that
 $\blacktriangleright HOD^{L(A,\mathbb{R})} \models GCH$

- It seems quite plausible that this is the test question for solving the general HOD-analysis problem of AD⁺.
 - This is the problem of showing that HOD is a "fine-structure model".
 - This problem in turn is deeply connected to the inner model problem for supercompact cardinals.
- But 6 months ago I would have said that any proof of the following theorem would also require the HOD-analysis.

Theorem

Assume $A \subseteq \mathbb{R}$ and that $L(A, \mathbb{R}) \models AD^+$. Then $\blacktriangleright HOD^{L(A,\mathbb{R})} \models Ultrapower Axiom.$

Background: Goldberg's Ultrapower Axiom

(Notation)

Suppose that $N \models \text{ZFC}$ is an inner model of ZFC, $U \in N$ and $N \models "U$ is a countable complete ultrafilter"

- ▶ N_U denotes the transitive collapse of Ult(N, U)
- ▶ $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Background: Goldberg's Ultrapower Axiom

(Notation)

Suppose that $N \models \text{ZFC}$ is an inner model of ZFC, $U \in N$ and $N \models "U$ is a countable complete ultrafilter"

- ▶ N_U denotes the transitive collapse of Ult(N, U)
- ▶ $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Definition (The Ultrapower Axiom)

Suppose that U and W are countably complete ultrafilters. Then there exist $W^* \in V_U$ and $U^* \in V_W$ such that the following hold. (1) $V_U \models "W^*$ is a countable complete ultrafilter". (2) $V_W \models "U^*$ is a countable complete ultrafilter".

Background: Goldberg's Ultrapower Axiom

(Notation)

Suppose that $N \models \text{ZFC}$ is an inner model of ZFC, $U \in N$ and $N \models "U$ is a countable complete ultrafilter"

- ▶ N_U denotes the transitive collapse of Ult(N, U)
- ▶ $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Definition (The Ultrapower Axiom)

Suppose that U and W are countably complete ultrafilters. Then there exist $W^* \in V_U$ and $U^* \in V_W$ such that the following hold.

- (1) $V_U \models "W^*$ is a countable complete ultrafilter".
- (2) $V_W \models "U^*$ is a countable complete ultrafilter".
- (3) $(V_U)_{W^*} = (V_W)_{U^*}$.

(4)
$$j_{W^*}^{V_U} \circ j_U^V = j_{U^*}^{V_W} \circ j_W^V.$$

Background: Consequences of the Ultrapower Axiom

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is strongly compact. Then one of the following hold.

- 1. κ is a supercompact cardinal.
- 2. κ is a measurable limit of supercompact cardinals.

This is the best possible result by the Menas Theorem.

Background: Consequences of the Ultrapower Axiom

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is strongly compact. Then one of the following hold.

- 1. κ is a supercompact cardinal.
- 2. κ is a measurable limit of supercompact cardinals.

This is the best possible result by the Menas Theorem.

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is supercompact. Then the following hold.

1. $2^{\gamma} = \gamma^+$ for all $\gamma \ge \kappa$.

Background: Consequences of the Ultrapower Axiom

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is strongly compact. Then one of the following hold.

- 1. κ is a supercompact cardinal.
- 2. κ is a measurable limit of supercompact cardinals.
- This is the best possible result by the Menas Theorem.

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is supercompact. Then the following hold.

- 1. $2^{\gamma} = \gamma^+$ for all $\gamma \ge \kappa$.
- 2. Suppose $A \subset \kappa$ and that A codes V_{κ} . Then $V = HOD_A$.

► V is a generic extension of HOD.

Does V = Ultimate-L vastly accelerate consistency strength?

Problem

Does ZFC + "V = Ultimate-L" together with

"There is a Woodin limit of Woodin cardinals"

prove the consistency of

▶ ZFC + " There is an Axiom I₀ cardinal"?

Does V = Ultimate-L vastly accelerate consistency strength?

Problem

Does ZFC + "V = Ultimate-L" together with

"There is a Woodin limit of Woodin cardinals"

```
prove the consistency of
```

▶ ZFC + " There is an Axiom I₀ cardinal"?

If not, what about:

Problem

Does ZFC + "V = Ultimate-L" together with

"There is a cardinal which is Mahlo to Woodin cardinals"

prove the consistency of

▶ ZFC + " There is an Axiom I₀ cardinal"?

Background: Exotic ZF models?

Theorem (AD^+)

Suppose $V = L(A, \mathbb{R})$ for some $A \subset \mathbb{R}$. Let

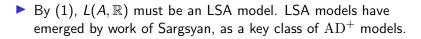
• Δ be the set of all $B \subset \mathbb{R}$ such that B is Suslin and co-Suslin

•
$$N = HOD_I$$
 where $I = Ord^{\omega}$.

Suppose that

- 1. Θ is a limit of Woodin cardinals in HOD.
- 2. The HOD-analysis holds in $L(\Delta)$.

Then $N_{\Theta} \models \mathbb{ZF}$.



Background: Exotic ZF models?

Theorem (AD^+)

Suppose $V = L(A, \mathbb{R})$ for some $A \subset \mathbb{R}$. Let

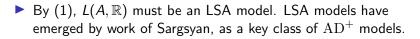
• Δ be the set of all $B \subset \mathbb{R}$ such that B is Suslin and co-Suslin

•
$$N = HOD_I$$
 where $I = Ord^{\omega}$.

Suppose that

- 1. Θ is a limit of Woodin cardinals in HOD.
- 2. The HOD-analysis holds in $L(\Delta)$.

Then $N_{\Theta} \models \mathbb{ZF}$.



N_⊖ is the first nontrivial candidate for a ZF model which (structurally) generalizes AD to **all** the levels of model.

Background: Exotic ZF models?

Theorem (AD^+)

Suppose $V = L(A, \mathbb{R})$ for some $A \subset \mathbb{R}$. Let

• Δ be the set of all $B \subset \mathbb{R}$ such that B is Suslin and co-Suslin

•
$$N = HOD_I$$
 where $I = Ord^{\omega}$.

Suppose that

- 1. Θ is a limit of Woodin cardinals in HOD.
- 2. The HOD-analysis holds in $L(\Delta)$.

Then $N_{\Theta} \models \operatorname{ZF}$.

▶ By (1), L(A, ℝ) must be an LSA model. LSA models have emerged by work of Sargsyan, as a key class of AD⁺ models.

- N_⊖ is the first nontrivial candidate for a ZF model which (structurally) generalizes AD to **all** the levels of model.
- Can (or must) N_{Θ} have an Axiom I₀ cardinal?

Or even rank initial segments with super-Reinhardt cardinals"