Exotic Models

Exotic categorical wellfounded models

Problem

Suppose $\mathrm{ZFC} + \varphi$ has a unique transitive model M. Must

- $M \models$ "$V = L$"?

Exotic categorical wellfounded models

Problem

Suppose $\mathrm{ZFC} + \varphi$ has a unique transitive model M. Must

- $M \models$ "$V = L$"?

Theorem (joint with Koellner)

Assume there is a Woodin cardinal with an inaccessible above. There is a formula φ such that the following holds.

- *For all $x \in \mathbb{R}$, if $M \models \mathrm{ZFC} + \varphi[x]$ and M is transitive, then M is unique and*
  - $M \models$ "$V \neq L[x]$".

Exotic categorical wellfounded models

Problem

Suppose $\mathrm{ZFC} + \varphi$ has a unique transitive model M. Must

- $M \models$ "$V = L$"?

Theorem (joint with Koellner)

Assume there is a Woodin cardinal with an inaccessible above. There is a formula φ such that the following holds.

- *For all $x \in \mathbb{R}$, if $M \models \mathrm{ZFC} + \varphi[x]$ and M is transitive, then M is unique and*
  - $M \models$ "$V \neq L[x]$".
- *For a Turing cone of x, there is transitive set M such that*
  - $M \models \mathrm{ZFC} + \varphi[x]$.

Exotic categorical wellfounded models

Problem

Suppose $\mathrm{ZFC} + \varphi$ has a unique transitive model M. Must

- $M \models$ "$V = L$"?

Theorem (joint with Koellner)

Assume there is a Woodin cardinal with an inaccessible above. There is a formula φ such that the following holds.

- *For all $x \in \mathbb{R}$, if $M \models \mathrm{ZFC} + \varphi[x]$ and M is transitive, then M is unique and*
  - $M \models$ "$V \neq L[x]$".
- *For a Turing cone of x, there is transitive set M such that*
  - $M \models \mathrm{ZFC} + \varphi[x]$.

- The proof is nowhere close to solving the problem.

Exotic MM^{++} models

Problem

Are any of the following consistent with MM^{++} and the existence of an extendible cardinal?

1. $\text{Axiom } (*)^{++}$.
2. $\mathrm{HOD} \models$ "$V = \text{Ultimate-}L$".
3. There is no inner model N of ZFC with the ω_2-approximation property and the ω_2-cover property such that ω_2 is strongly inaccessible in N.

Exotic MM^{++} models

Problem

Are any of the following consistent with MM^{++} and the existence of an extendible cardinal?

1. $\mathrm{Axiom}\ (*)^{++}$.
2. $\mathrm{HOD} \models$ "$V = \mathrm{Ultimate}\text{-}L$".
3. There is no inner model N of ZFC with the ω_2-approximation property and the ω_2-cover property such that ω_2 is strongly inaccessible in N.

Theorem

Suppose that δ is a super-Reinhardt cardinal. Then there is a generic extension of $V[G]$ of V such that the following that

- $V[G]_\delta \models \mathrm{ZFC} + \mathrm{MM}^{++} +$ "There is an extendible cardinal"

and such that (3) *holds in $V[G]_\delta$.*

- But maybe the theorem is vacuously true.

Revised HOD Conjecture

- If super-Reinhardt cardinals are consistent then:
  - The HOD Conjecture is false.

Problem (Revised HOD Conjecture)

Suppose that δ is an extendible cardinal. Show that one of the following **must** hold.

1. HOD is a weak extender model of δ is supercompact.
   - Equivalently, HOD is close to V as in
     - The HOD Dichotomy Theorem.
2. HOD has no supercompact cardinals.

The HOD-analysis from AD^+-theory

Problem

Assume $A \subseteq \mathbb{R}$ and that $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Show that

- $\mathrm{HOD}^{L(A,\mathbb{R})} \models \mathrm{GCH}$.

- It seems quite plausible that this is the test question for solving the general HOD-analysis problem of AD^+.
  - This is the problem of showing that HOD is a "fine-structure model".
    - This problem in turn is deeply connected to the inner model problem for supercompact cardinals.

The HOD-analysis from AD^+-theory

Problem

Assume $A \subseteq \mathbb{R}$ and that $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Show that

- $\mathrm{HOD}^{L(A,\mathbb{R})} \models \mathrm{GCH}$.

- It seems quite plausible that this is the test question for solving the general HOD-analysis problem of AD^+.
  - This is the problem of showing that HOD is a "fine-structure model".
    - This problem in turn is deeply connected to the inner model problem for supercompact cardinals.

- But 6 months ago I would have said that any proof of the following theorem would also require the HOD-analysis.

Theorem

Assume $A \subseteq \mathbb{R}$ and that $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Then

- $\mathrm{HOD}^{L(A,\mathbb{R})} \models \mathrm{Ultrapower\ Axiom}$.

Background: Goldberg's Ultrapower Axiom

(Notation)

Suppose that $N \models \mathrm{ZFC}$ is an inner model of ZFC, $U \in N$ and

$$N \models \text{“}U \text{ is a countable complete ultrafilter”}$$

- N_U denotes the transitive collapse of $\mathrm{Ult}(N, U)$
- $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Background: Goldberg's Ultrapower Axiom

(Notation)

Suppose that $N \models \mathrm{ZFC}$ is an inner model of ZFC, $U \in N$ and

$$N \models \text{"}U \text{ is a countable complete ultrafilter"}$$

- N_U denotes the transitive collapse of $\mathrm{Ult}(N, U)$
- $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Definition (The Ultrapower Axiom)

Suppose that U and W are countably complete ultrafilters. Then there exist $W^* \in V_U$ and $U^* \in V_W$ such that the following hold.

(1) $V_U \models$ "W^* is a countable complete ultrafilter".

(2) $V_W \models$ "U^* is a countable complete ultrafilter".

Background: Goldberg's Ultrapower Axiom

(Notation)

Suppose that $N \models \mathrm{ZFC}$ is an inner model of ZFC, $U \in N$ and

$$N \models \text{“}U \text{ is a countable complete ultrafilter”}$$

- N_U denotes the transitive collapse of $\mathrm{Ult}(N, U)$
- $j_U^N : N \to N_U$ denotes the associated ultrapower embedding.

Definition (The Ultrapower Axiom)

Suppose that U and W are countably complete ultrafilters. Then there exist $W^* \in V_U$ and $U^* \in V_W$ such that the following hold.

(1) $V_U \models \text{“}W^* \text{ is a countable complete ultrafilter”}$.

(2) $V_W \models \text{“}U^* \text{ is a countable complete ultrafilter”}$.

(3) $(V_U)_{W^*} = (V_W)_{U^*}$.

(4) $j_{W^*}^{V_U} \circ j_U^V = j_{U^*}^{V_W} \circ j_W^V$.

Background: Consequences of the Ultrapower Axiom

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is strongly compact. Then one of the following hold.

1. *κ is a supercompact cardinal.*
2. *κ is a measurable limit of supercompact cardinals.*

- This is the best possible result by the Menas Theorem.

Background: Consequences of the Ultrapower Axiom

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is strongly compact. Then one of the following hold.

1. *κ is a supercompact cardinal.*
2. *κ is a measurable limit of supercompact cardinals.*

- This is the best possible result by the Menas Theorem.

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is supercompact. Then the following hold.

1. $2^\gamma = \gamma^+$ *for all* $\gamma \geq \kappa$.

Background: Consequences of the Ultrapower Axiom

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is strongly compact. Then one of the following hold.

1. *κ is a supercompact cardinal.*
2. *κ is a measurable limit of supercompact cardinals.*

- This is the best possible result by the Menas Theorem.

Theorem (Goldberg)

Assume the Ultrapower Axiom and that κ is supercompact. Then the following hold.

1. $2^\gamma = \gamma^+$ *for all* $\gamma \geq \kappa$.
2. *Suppose $A \subset \kappa$ and that A codes V_κ. Then $V = \mathrm{HOD}_A$.*
   - *V is a generic extension of HOD.*

Does $V = \text{Ultimate-}L$ vastly accelerate consistency strength?

Problem

Does ZFC + "$V = \text{Ultimate-}L$" together with

- "There is a Woodin limit of Woodin cardinals"

prove the consistency of

- ZFC + " There is an Axiom I_0 cardinal"?

Does $V = \text{Ultimate-}L$ vastly accelerate consistency strength?

Problem

Does $\mathrm{ZFC} +$ "$V = \text{Ultimate-}L$" together with

- "There is a Woodin limit of Woodin cardinals"

prove the consistency of

- $\mathrm{ZFC} +$ " There is an Axiom I_0 cardinal"?

If not, what about:

Problem

Does $\mathrm{ZFC} +$ "$V = \text{Ultimate-}L$" together with

- "There is a cardinal which is Mahlo to Woodin cardinals"

prove the consistency of

- $\mathrm{ZFC} +$ " There is an Axiom I_0 cardinal"?

Background: Exotic ZF models?

Theorem (AD^+)

Suppose $V = L(A, \mathbb{R})$ for some $A \subset \mathbb{R}$. Let

- *Δ be the set of all $B \subset \mathbb{R}$ such that B is Suslin and co-Suslin*
- *$N = \mathrm{HOD}_I$ where $I = \mathrm{Ord}^\omega$.*

Suppose that

1. *Θ is a limit of Woodin cardinals in HOD.*
2. *The HOD-analysis holds in $L(\Delta)$.*

Then $N_\Theta \models \mathrm{ZF}$.

- By (1), $L(A, \mathbb{R})$ must be an LSA model. LSA models have emerged by work of Sargsyan, as a key class of AD^+ models.

Background: Exotic ZF models?

Theorem (AD^+)

Suppose $V = L(A, \mathbb{R})$ for some $A \subset \mathbb{R}$. Let

- *Δ be the set of all $B \subset \mathbb{R}$ such that B is Suslin and co-Suslin*
- *$N = \mathrm{HOD}_I$ where $I = \mathrm{Ord}^\omega$.*

Suppose that

1. *Θ is a limit of Woodin cardinals in HOD.*
2. *The HOD-analysis holds in $L(\Delta)$.*

Then $N_\Theta \models \mathrm{ZF}$.

- By (1), $L(A, \mathbb{R})$ must be an LSA model. LSA models have emerged by work of Sargsyan, as a key class of AD^+ models.

- N_Θ is the first nontrivial candidate for a ZF model which (structurally) generalizes AD to **all** the levels of model.

Background: Exotic ZF models?

Theorem (AD^+)

Suppose $V = L(A, \mathbb{R})$ for some $A \subset \mathbb{R}$. Let

- *Δ be the set of all $B \subset \mathbb{R}$ such that B is Suslin and co-Suslin*
- *$N = \mathrm{HOD}_I$ where $I = \mathrm{Ord}^\omega$.*

Suppose that

1. *Θ is a limit of Woodin cardinals in HOD.*
2. *The HOD-analysis holds in $L(\Delta)$.*

Then $N_\Theta \models \mathrm{ZF}$.

- By (1), $L(A, \mathbb{R})$ must be an LSA model. LSA models have emerged by work of Sargsyan, as a key class of AD^+ models.

- N_Θ is the first nontrivial candidate for a ZF model which (structurally) generalizes AD to **all** the levels of model.
- Can (or must) N_Θ have an Axiom I_0 cardinal?
  - Or even rank initial segments with super-Reinhardt cardinals"