What Model companionship can say about the Continuum problem

Matteo Viale

Dipartimento di Matematica
Università di Torino

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Section 1

Forcing axioms

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

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The talk is mainly aimed at formulating precisely the second of these two concepts.

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- MM^{++} and $(*)$ are forcing axioms for $\aleph_1$ the first uncountable cardinal.
- Baire's category theorem is a "topological" forcing axiom for $\aleph_0$.
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Section 2

Algebraic closure and model companionship

Algebraic closure of structures for $\{+, \cdot, 0, 1\}$

Structures	Axioms	Example
Commutative semirings with no zero divisors	$\forall x, y\, (x \cdot y = y \cdot x)$ $\forall x, y, z\, [(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ $\forall x\, (x \cdot 1 = x \wedge 1 \cdot x = x)$ $\forall x, y\, (x + y = y + x)$ $\forall x, y, z\, [(x + y) + z = x + (y + z)]$ $\forall y\, (x + 0 = x \wedge 0 + x = x)$ $\forall x, y, z\, [(x + y) \cdot z = (x \cdot y) + (x \cdot z)]$ $\forall x, y\, [x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)]$	$\mathbb{N}$
Integral domains	$\forall x \exists y\, (x + y = 0)$	$\mathbb{Z}$
Fields	$\forall x\, [x \neq 0 \rightarrow \exists y\, (x \cdot y = 1)]$	$\mathbb{Q}$
Algebraically closed fields	for all $n \geq 1$ $\forall x_0 \ldots x_n \exists y \sum x_i \cdot y^i = 0$	$\mathbb{C}$

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Existentially closed structures and model companionship

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$$\exists x\, (x^2 - 2 = 0)? \qquad \exists x\, (x^3 + 2x + i = 0)?$$

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Definition

Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

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Example

In the vocabulary $\{+, \cdot, 0, 1\}$, the atomic formulae are diophantine equations and the quantifier free formulae with parameters in a ring $\mathcal{M}$ define the constructible sets (in the sense of algebraic geometry) of $\mathcal{M}$:

$$\bigvee_{j=1}^{l} \left[\bigwedge_{i=1}^{k_j} p_{ij}(a_1^{ij}, \ldots, a_{m_{ij}}^{ij}, x_1, \ldots, x_n) = 0 \wedge \bigwedge_{d=1}^{m_j} \neg q_{dj}(b_1^{dj}, \ldots, b_{k_{dj}}^{dj}, x_1, \ldots, x_n) = 0 \right]$$

with each a_k^{ij}, b_k^{dj} elements of $\mathcal{M}$ and $p_{ij}(y_1, \ldots, y_{m_{ij}}, x_1, \ldots, x_n) = 0$, $q_{dj}(z_1, \ldots, z_{k_{dj}}, x_1, \ldots, x_n) = 0$ diophantine equations (of degree 1 in the y_l, z_h-s).

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- A τ-formula $\psi(x_0, \dots, x_n)$ is a Σ_1-formula if it is of the form $\exists y_0, \dots, y_k \phi(y_0, \dots, y_k, x_0, \dots, x_n)$ with $\phi(y_0, \dots, y_k, x_0, \dots, x_n)$ quantifier free.

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Given a τ-theory S, a τ-structure $\mathcal{M}$ is S-ec if:

- there is a model of S $\mathcal{N} \sqsupseteq \mathcal{M}$,
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The $\{+,\cdot,0,1\}$-theory of integral domains has the $\{+,\cdot,0,1\}$-theory of algebraically closed fields as its model companion.

The right vocabulary for a mathematical theory

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Consider Group Theory

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Axioms of groups in $\{\cdot, e\}$

$$\forall x, y, z\,[(x \cdot y) \cdot z = x \cdot (y \cdot z)],$$
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Axioms of groups in $\{R, e\}$ with R a ternary relation symbol

$\forall x, y \exists! z\, R(x, y, z)$,

$\forall x, y, z, w, t\,[((R(x, y, w) \wedge R(y, z, t)) \rightarrow \exists u\,(R(x, t, u) \wedge R(w, z, u))]$,

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$\forall y\,[R(e, y, y) \wedge R(y, e, y)]$,

$\forall x \exists y\,[R(x, y, e) \wedge R(y, x, e)]$.

The two axiomatizions are equivalent in the vocabulary $\{R, \cdot, e\}$, modulo the axiom

$$\forall x, y, z\,(R(x, y, z) \leftrightarrow x \cdot y = z)$$

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Formalizing in the $\{\in\}$-vocabulary the notion of ordered pair:

Kuratowski's trick: $\langle y, z\rangle$ is coded in set theory by the set $\{\{y\}, \{y, z\}\}$.

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Formalizing in the $\{\in\}$-vocabulary the notion of ordered pair:

Kuratowski's trick: $\langle y, z\rangle$ is coded in set theory by the set $\{\{y\}, \{y, z\}\}$.
In set theory the standard $\in$-formula expressing $x = \langle y, z\rangle$ is

$$\exists t \exists u\, [\forall w\, (w \in x \leftrightarrow w = t \vee w = u) \wedge \forall v\, (v \in t \leftrightarrow v = y) \wedge \forall v\, (v \in u \leftrightarrow v = y \vee v = z)].$$

The right vocabulary for set theory

The vocabulary $\in_{\Delta_0}$ for set theory

- constants for $\emptyset, \mathbb{N}$,
- relation symbols R_ϕ for any lightface Δ_0-property $\phi(x_1, \ldots, x_n)$,
- function symbols for a finite list of basic set theoretic constructors.

The right vocabulary for set theory

Lightface Δ_0-properties

- $\{R \in V : R$ is an n-ary relation$\}$,
- $\{f \in V : f$ is a function$\}$,
- $\left\{\langle a, b\rangle \in V^2 : a \subseteq b\right\}$,
- $\ldots$
- $\{\langle a_1, \ldots, a_n\rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$ for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set.

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The *lightface Δ_0-properties* are those described in the last item above and include all those listed in some of the above items.

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Complicated set theoretic relations

- $\left\{\langle X, Y\rangle \in V^2 : |X| = |Y|\right\}$,
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- any relation which is not a Δ_1-property $(\Delta_0 \subseteq \Delta_1)$.

The right vocabulary for set theory

Basic set theoretic operations

- $\pi^n_j : \langle a_1, \ldots, a_n \rangle \mapsto a_j$,
- $\langle X, Y \rangle \mapsto X \times Y$,
- $\langle X, Y \rangle \mapsto \{X, Y\}$,
- $\ldots$
- Any provably total function whose graph is a lightface Δ_0-property.

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Lightface Δ_0-properties

$$\{\langle a_1, \ldots, a_n\rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$$

for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set.

Basic set theoretic operations

Any total function whose graph is a lightface Δ_0-property.

Section 3

Formalization of set theory

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Notational convention: smallcase variables indicate sets, uppercase variables indicate classes.

Universal axioms

- **Extensionality:** $\forall X, Y\, [(X \subseteq Y \wedge Y \subseteq X) \leftrightarrow X = Y]$.
- **Comprehension:** $\forall X\, (\text{Set}(X) \leftrightarrow X \in V) \wedge \forall X\, (X \subseteq V)$.
- **Foundation:**

$$\forall F\, [(F \text{ is a function } \wedge \text{dom}(F) = \mathbb{N}) \rightarrow \exists n \in \mathbb{N}\, F(n+1) \notin F(n)].$$

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Existence Axioms:

- **Emptyset:** $(\forall x\, x \notin \emptyset) \wedge (\emptyset \in V)$,
- **Infinity:**
  $\text{Set}(\mathbb{N}) \wedge \forall x\, [x \in \mathbb{N} \leftrightarrow (x \text{ is a finite Von Neumann ordinal})]$.

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Basic construction principles:

- **Union and Pair:** $\forall X, Y, w\, [w \in X \cup Y \leftrightarrow (w \in X \vee w \in Y)], \ldots$
- **Separation:** $\forall P, x\, [(x \in V) \rightarrow (P \cap x) \in V]$.

Axioms of Morse-Kelley Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Strong construction principles:

- **Comprehension (b):** For every $\in_{\Delta_0}$-formula $\psi(\vec{x}, \vec{Y})$

$$\forall \vec{Y}\, \exists Z\, \forall x\, [x \in Z \leftrightarrow (x \in V \wedge \exists x_0, \ldots, x_n (x = \langle x_0, \ldots, x_n \rangle \wedge \psi(x_0, \ldots, x_n, \vec{Y})))].$$

- **Replacement:**

$$\forall F, x\, [(F \text{ is a function } \wedge (x \in V) \wedge (x \subseteq \text{dom}(F))) \rightarrow (F[x] \in V)].$$

- **Powerset:**

$$\forall x\, [(x \in V) \rightarrow [\forall z\, (z \in \mathcal{P}(X) \leftrightarrow z \subseteq x) \wedge \mathcal{P}(x) \in V]].$$

- **Choice:**

$$\begin{aligned}
&\forall F[\\
&\quad F \text{ is a function} \wedge \forall x\, (x \in \text{dom}(F) \rightarrow F(x) \neq \emptyset)\\
&\rightarrow\\
&\quad \exists G\, (G \text{ is a function} \wedge \text{dom}(G) = \text{dom}(F) \wedge \forall x\, (x \in \text{dom}(G) \rightarrow G(x) \in F(x))\\
&].
\end{aligned}$$

Section 4

Algebraic maximality for set theory

Existentially closed structures for set theory

Theorem (Levy)

Let κ be an infinite cardinal.
Then

$$\langle H_{\kappa^+}, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle \prec_1 \langle V, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

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Theorem (Shoenfield, 1961)

Let $V[G]$ be a forcing extension of V. Then

$$\langle H_{\aleph_1}, \in_{\Delta_0}\rangle \prec_1 \langle V[G], \in_{\Delta_0}\rangle.$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- (modulo a Borel isomorphism) $\mathbb{R} \approx \mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$ and UB is a family of subsets of $\mathcal{P}(\mathbb{N})$.
- Every univ. Baire set A of V can be naturally lifted to a univ. Baire set $A^{V[G]}$ of $V[G]$ for any forcing extension $V[G]$ of V.

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Theorem (Feng-Magidor-Woodin, 1992)

Let $V[G]$ be a forcing extension of V. Then

$$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathsf{UB}^V \rangle \prec_1 \langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathsf{UB}^V \rangle.$$

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Theorem (Woodin, 1985+Martin-Steel, 1989+ V.-Venturi, 2020)

Assume there is a proper class of Woodin's cardinals. Then the theory of

$$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$$

*is the **model companion** of the theory of*

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Algebraic maximality for $\mathcal{P}(\mathbb{N})$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is Σ_1-elementary in $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V
MK+ large cardinal axioms	The theory of $\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is the **model companion** of the theory of $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V

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- NS $\subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

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Definition

Let B be a cba. B is SSP if whenever $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V \rangle \sqsubseteq \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]} \rangle.$$

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Strong Bounded Martin's maximum BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

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MM^{++} is consistent with the existence of any axiom of large cardinals.

Applications of BMM^{++}

Assume BMM^{++}. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^+$.
  Todorčević, Mathematical Research Letters, 9(2), 2006.
- Whitehead's conjecture on free groups is false, (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}: There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
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UB-BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

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Theorem (Woodin)

MM^{++} *implies* UB-BMM^{++}*.*

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$(*)_{\mathrm{UB}}$ is a natural strengthening of Woodin's axiom $(*)$.

Theorem (Asperó-Schindler)

Assume there is a proper class of Woodin cardinals. Then $(*)_{\mathrm{UB}}$ *if and only if* UB-BMM^{++}.

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$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \text{ is in } \mathcal{P}(\mathbb{R})^{L(\mathbb{R})^V} \rangle$$

is Σ_1*-elementary in*

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Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

In signature $\in_{\Delta_0}$ $\neg$CH can be formalized by the Π_2-sentence in parameter $\aleph_1$ (the first uncountable ordinal/cardinal):

$$\forall f\,[\underbrace{f \text{ is a function}}_{\Delta_0(f)} \wedge \underbrace{\mathrm{dom}(f)=\aleph_1}_{\Delta_0(f,\aleph_1)}) \rightarrow \exists r\,(\underbrace{r\subseteq\mathbb{N}}_{\Delta_0(r,\mathbb{N})} \wedge \underbrace{r\notin\mathrm{ran}(f)}_{\Delta_0(r,f)})]$$

Note that $\aleph_1 \in H_{\aleph_2}$.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

Theorem (Woodin)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* $\mathrm{UB\text{-}BMM}^{++}$*).*
- *For any* Π_2*-sentences* ψ *for* $\in_{\Delta_0} \cup \{\aleph_1, \mathrm{NS}\} \cup \left\{A : A \in \mathrm{UB}^V\right\}$

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle \models \psi$$

if and only if

ψ ***is true in*** $H_{\aleph_2}^{V[G]}$ ***for some forcing extension*** $V[G]$ ***of*** V.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

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$$\langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \models \psi$$

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Sealing can be removed by replacing UB with $\mathcal{P}(\mathbb{R})^N$ for some nice inner model N of determinacy in the formulation of BMM^{*++} and in the relevant spots.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume Vopenka's principle, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *The theory T of the structure*

$$\mathcal{M} = \langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$$

*is the **model companion** of the theory S of the structure*

$$\langle V, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle.$$

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$$\langle V, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle.$$

- *Letting $S_{\forall\vee\exists}$ be the boolean combination of existential sentences which are in S, and ψ be a Π_2-sentence,*
  $\mathcal{M}$ ***models*** ψ ***if and only*** $\psi + S_{\forall\vee\exists}$ ***is consistent***.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

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if and only if

ψ *is true in* $H_{\aleph_2}^{V[G]}$ *for some forcing extension* $V[G]$ *of* V*.*

if and only if

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where S *is the theory of the structure*

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Sealing can be removed if one replaces UB^V with $\mathcal{P}(R)^{L(\text{Ord}^{\mathbb{N}})}$ in the formulation of BMM^{*++} and in the relevant spots.

Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$ is a *substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ forcing axioms	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$ is a Σ_1-*substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ large cardinal axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ have the same **model companion** theory
MK+ large cardinals + forcing axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ have as **model companion** the theory of $\langle H^V_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V, A^V : A \in \mathrm{UB}^V \rangle$

Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$ is a *substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
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Section 5

Appendixes

Appendix 0: Some references

A few surveys on Gödel's program and the Continuum problem:

- J. Bagaria, *Natural axioms on set theory and the continuum problem*, CRM Preprint, 591, 2004.
- P. Koellner, *On the question of absolute undecidability*, in *Kurt Gödel: essays for his centennial*, Lect. Notes Log. 33, 2010.
- G. Venturi and M. Viale, *What model companionship can say about the Continuum problem*, arXiv:2204.13756, 2022.
- M. Viale, *Strong forcing axioms and the continuum problem*, in *Séminaire Bourbaki. Volume 2022/2023. Exposés 1197–1211*, 2023, (SMF).
- W. H. Woodin, *The Continuum hypothesis Part I*, Notices of AMS, 48(6), 2001.
- W. H. Woodin, *The Continuum hypothesis Part II*, Notices of AMS, 48(7), 2001.

Appendix 1: Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.

Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.

Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.

This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.

Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

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Appendix 2: Sealing

Definition (Woodin)

Given $(\mathcal{D}, W, \in_{\Delta_0})$ transitive model of MK, let N^W be the set $\mathcal{P}(H_{\aleph_1})^{L(\mathrm{UB})^W}$, where $L(\mathrm{UB})^W$ is the smallest transitive model of ZF containing UB^W.

(A weak form of) Sealing holds in a model $(\mathcal{C}, V, \in_{\Delta_0})$ of MK+*enough large cardinals* if whenever $V[G]$ is a forcing extension of V and $V[H]$ a forcing extension of $V[G]$ we have that

$$(N^{V[G]}, H_{\aleph_1}^{V[G]}, \in_{\Delta_0}) \prec (N^{V[H]}, H_{\aleph_1}^{V[H]}, \in_{\Delta_0}).$$

Theorem (Woodin)

Assume V models κ is supercompact and there are class many Woodin cardinals. Let $V[H]$ be a generic extension of V where κ is countable. Then sealing holds in $V[H]$.

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